Stackelberg oligopoly TU-games: characterization of the core and 1-concavity of the dual game

Theo Driessen* Dongshuang Hou† Aymeric Lardon‡

July 22, 2011

Abstract

In this article we consider Stackelberg oligopoly TU-games in γ-characteristic function form (Chander and Tulkens 1997) in which any deviating coalition produces an output at a first period as a leader and outsiders simultaneously and independently play a quantity at a second period as followers. We assume that the inverse demand function is linear and that firms operate at constant but possibly distinct marginal costs. Generally speaking, for any TU-game we show that the 1-concavity property of its dual game is a necessary and sufficient condition under which the core of the initial game is non-empty and coincides with the set of imputations. The dual game of a Stackelberg oligopoly TU-game is of great interest since it describes the marginal contribution of followers to join the grand coalition by turning leaders. The aim is to provide a necessary and sufficient condition which ensures that the dual game of a Stackelberg oligopoly TU-game satisfies the 1-concavity property. Moreover, we prove that this condition depends on the heterogeneity of firms' marginal costs, i.e., the dual game is 1-concave if and only if firms' marginal costs are not too heterogeneous. This last result extends Marini and Currarini's core non-emptiness result (2003) for oligopoly situations.

Keywords: Stackelberg oligopoly TU-game; Dual game; 1-concavity

1 Introduction

Usually, oligopoly situations are modeled by means of non-cooperative games. Every profit-maximizing firm pursues Nash strategies and the resulting outcome is not

*T.S.H. Driessen, Faculty of Electrical Engineering, Mathematics and Computer Science, Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands, e-mail: t.s.h.driessen@ewi.utwente.nl Webpage: http://dmmp.ewi.utwente.nl/people/staff/driessen

†Dongshuang Hou, University of Twente, Enschede, The Netherlands, e-mail: dshhou@126.com

‡A. LARDON, University of Saint-Etienne, GATE Lyon Saint-Etienne, CNRS, 6 rue Basse des Rives, 42023 Saint-Etienne, France, e-mail: aymeric.lardon@univ-st-etienne.fr, Tel: (+33)(0)4.77.42.19.62, Fax: (+33)(0)4.77.42.19.50

Pareto optimal. Yet, it is known that firms are better off by forming cartels and that Pareto efficiency is achieved when all the firms merge together. A problem faced by the members of a cartel is the stability of the agreement and non-cooperative game theory predicts that the cartel members always have an incentive to deviate from the agreed-upon output decision.

However, in some oligopoly situations firms don't always behave non-cooperatively and if sufficient communication is feasible it may be possible for firms to sign agreements. A question is then whether it is possible for firms to agree all together and coordinate their decision to achieve Pareto efficiency. For that, we consider a fully cooperative approach for oligopoly situations. Under this approach, firms are allowed to sign binding agreements in order to form cartels called coalitions. Under such an assumption cooperative games called oligopoly TU(Transferable Utility)-games can be defined and the existence of stable collusive behaviors is then related to the non-emptiness of the core of such games. Aumann (1959) proposes two approaches in order to define cooperative games: according to the first, every cartel computes the total profit which it can guarantee itself regardless of what outsiders do; the second approach consists in computing the minimal profit for which outsiders can prevent the cartel members from getting more. These two assumptions lead to consider the α and β-characteristic functions respectively. However, these two approaches can be questioned since outsiders probably cause substantial damages upon themselves by minimizing the profit of the cartel. This is why Chander and Tulkens (1997) propose an alternative blocking rule where external firms choose their strategy individually as a best reply to the cartel action. This leads to consider the "partial agreement characteristic function" or, for short, the γ-characteristic function.

As regards Cournot oligopoly TU-games in α and β-characteristic function forms with or without transferable technologies,[1] Zhao (1999a,b) shows that the α and β-characteristic functions lead to the same set of Cournot oligopoly TU-games. When technologies are transferable, Zhao (1999a) provides a necessary and sufficient condition to establish the convexity property in case the inverse demand function and cost functions are linear. Although these games may fail to be convex in general, Norde et al. (2002) show they are nevertheless totally balanced. When technologies are not transferable, Zhao (1999b) proves that the core of such games is non-empty if every individual profit function is continuous and concave.[2] Furthermore, Norde et al. (2002) show that these games are convex in case the inverse demand function and cost functions are linear, and Driessen and Meinhardt (2005) provide economically meaningful sufficient conditions to guarantee the convexity property in a more general case.

[1] We refer to Norde et al. (2002) for a detailed discussion of this distinction.

[2] Zhao shows that the core is non-empty for general TU-games in β-characteristic function form in which every strategy set is compact and convex, every utility function is continuous and concave, and satisfying the strong separability condition that requires that the utility function of a coalition and each of its members' utility functions have the same minimizers. Zhao proves that Cournot oligopoly TU-games satisfy this latter condition.

As regards Cournot oligopoly TU-games in γ-characteristic function form without transferable technologies, Lardon (2009) shows that the differentiability of the inverse demand function ensures that these games are well-defined and provides two core non-emptiness results. The first result establishes that such games are balanced, and therefore have a non-empty core, if every individual profit function is continuous and concave. When cost functions are linear, the second result provides a single-valued allocation rule in the core, called NP(Nash Pro rata)-value, which is characterized by four axioms: efficiency, null firm, monotonicity and non-cooperative fairness.[3] In a more general framework, Lardon (2010) shows that if the inverse demand function is continuous but not necessarily differentiable, it is always possible to define a Cournot oligopoly interval game[4] in γ-characteristic function form. This set of Cournot oligopoly interval games includes the set of Cournot oligopoly TU-games in γ-characteristic function form. Lardon considers two extensions of the core, the interval core and the standard core, and provides a necessary and sufficient condition for the non-emptiness of each of these two core solution concepts.
For general TU-games, Marini and Currarini (2003) associate a two-stage structure with the γ-characteristic function. In this temporal sequence every deviating coalition possesses a first-mover advantage by acting as a leader while outsiders play their individual best reply strategies as followers. Under standard assumptions on the payoff functions,[5] they prove that if the players and externalities are symmetric[6] and the game has strategic complementarities then the equal division solution belongs to the core. They also study a quantity competition setting where strategies are substitutes for which they show that the same mechanism underlying their core result determines the non-emptiness of the core.

Marini and Currarini's core non-emptiness result (2003) raises two questions. The first concerns the core structure of TU-games in sequential form since they only provide a single-valued allocation rule (the equal division solution) in the core. The second turns on the role of the symmetric players assumption on the non-emptiness of the core. In this article we answer both questions by considering the two-stage structure associated with the γ-characteristic function as suggested by Marini and Currarini (2003) for oligopoly situations. The introduction of this temporal sequence leads to consider a specific set of cooperative oligopoly games, i.e., the set of Stackelberg oligopoly TU-games in γ-characteristic function form. Thus, contrary to Cournot oligopoly TU-games in γ-characteristic function form in which all the firms simultaneously choose their strategies, every deviating coalition produces an output at a first period as a leader and outsiders simultaneously and independently play a quantity at a second period as followers. We assume that the

[3] We refer to Lardon (2009) for a precise description of these axioms.

[4] Interval games are introduced by Branzei et al. (2003).

[5] Every player's payoff function is assumed twice differentiable and strictly concave on its own strategy set.

[6] The players are symmetric if they have identical payoff functions and strategy sets. Externalities are symmetric if they are either positive or negative.

inverse demand function is linear and firms operate at constant but possibly distinct marginal costs. Thus, contrary to Marini and Currarini (2003) the payoff (profit) functions are not necessarily identical. The 1-concavity property of the dual game of a Stackelberg oligopoly TU-game answers the above two questions raised by Marini and Currarini's core non-emptiness result (2003). Indeed, generally speaking, for any TU-game we show that the 1-concavity property of its dual game is a necessary and sufficient condition under which the core of the initial game is non-empty and coincides with the set of imputations. Particularly, the nucleolus of such a 1-concave dual game agrees with the center of gravity of the set of imputations. The dual game of a Stackelberg oligopoly TU-game is of great interest since it gives the marginal contribution of followers to join the grand coalition by turning leaders. In order to establish the 1-concavity of the dual game we first characterize the core by proving that it is equal to the set of imputations which answers the first question. The reason is that the first-mover advantage gives too much power to singletons so that the worth of every deviating coalition is less than or equal to the sum of its members' individual worths except for the grand coalition. Then, we provide a necessary and sufficient condition under which the core is non-empty. Finally, we prove that this condition depends on the heterogeneity of firms' marginal costs, i.e., for a fixed number of firms the dual game is 1-concave if and only if firms' marginal costs are not too heterogeneous. The more the number of firms, the less the heterogeneity of firms' marginal costs must be in order to ensure the 1-concavity of the dual game, and so the non-emptiness of the core, which answers the second question. Surprisingly, in case the inverse demand function is strictly concave, we provide an example in which the opposite result holds, i.e., when the heterogeneity of firms' marginal costs increases the core becomes larger.

The remainder of this article is structured as follows. In section 2 we recall some basic definitions in cooperative game theory and show that Stackelberg oligopoly TU-games in γ-characteristic function form are well-defined. The aim of Section 3 is to establish the 1-concavity of the dual game. Section 4 gives some concluding remarks.

2 The model

Before introducing Stackelberg oligopoly TU-games in γ-characteristic function form (Chander and Tulkens 1997), we recall some basic definitions in cooperative game theory. Given a set of players N, we call a subset $S \in 2^N \backslash \{\emptyset\}$, a **coalition**. The **size** $s = |S|$ of coalition S is the number of players in S. A **TU-game** (N, v) is a **set function** $v : 2^N \longrightarrow \mathbb{R}$ with the convention that $v(\emptyset) = 0$, which assigns a number $v(S) \in \mathbb{R}$ to every coalition $S \in 2^N \backslash \{\emptyset\}$. The number $v(S)$ is the worth of coalition S and represents the maximal amount of monetary benefits due to the mutual cooperation among the members of the coalition, on the understanding that there are no benefits by absence of players. We denote by G the **set of TU-games** where (N, v) is a representative element of G.

With every TU-game $(N, v) \in G$ there is associated its dual game (N, v^*) defined by $v^*(S) = v(N) - v(N \backslash S)$ for all $S \in 2^N$. That is, the worth of any coalition in the dual game is given by the coalitionally marginal contribution with respect to the formation of the grand coalition N in the original game. Particularly, $v^*(\emptyset) = 0$, $v^*(N) = v(N)$, and so, the marginal contribution $m_i^{v^*} = v^*(N) - v^*(N \backslash \{i\}) = v(\{i\})$ for all $i \in N$.
In the framework of the division problem of the benefits $v(N)$ of the grand coalition N among the potential players, any allocation scheme of the form $x = (x_i)_{i \in N} \in \mathbb{R}^n$ is supposed to meet, besides the efficiency principle $\sum_{i \in N} x_i = v(N)$, the so-called individual rationality condition in that each player is allocated at least the individual worth, i.e., $x_i \geq v(\{i\})$ for all $i \in N$. For any TU-game $(N, v) \in G$, these two conditions lead to consider the set of imputations:

$$I(N, v) = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N) \text{ and } \forall i \in N,\, x_i \geq v(\{i\}) \right\}.$$

The best known set-valued solution concept called core requires the group rationality condition in that the aggregate allocation to the members of any coalition is at least its coalitional worth:

$$C(N, v) = \left\{ x \in I(N, v) : \forall S \in 2^N \backslash \{\emptyset, N\},\, \sum_{i \in S} x_i \geq v(S) \right\} \tag{1}$$

Of significant importance, is the upper core bound composed of the marginal contributions $m_i^v = v(N) - v(N \backslash \{i\})$, $i \in N$, with respect to the formation of the grand coalition N in the game (N, v). Obviously, $x_i \leq m_i^v$ for all $i \in N$ and all $x \in C(N, v)$. In this context, we focus on the following core catcher called Core-Cover:

$$CC(N, v) = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N) \text{ and } \forall i \in N,\, x_i \leq m_i^v \right\}.$$

Now, we introduce the Stackelberg oligopoly model. A **Stackelberg oligopoly situation** is a quintuplet $(L, F, (q_i, C_i)_{i \in N}, p)$ defined as:

1. the disjoint finite **sets of leaders and followers** L and F respectively where $L \cup F = \{1, 2, \ldots, n\}$ is the **set of firms** denoted by N;
2. for every $i \in N$, a **capacity constraint** $q_i \in \overline{\mathbb{R}}_+$;
3. for every $i \in N$, an **individual cost function** $C_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$;
4. an **inverse demand function** $p : \mathbb{R}_+ \longrightarrow \mathbb{R}$ which assigns to any aggregate quantity $X \in \mathbb{R}_+$ the unit price $p(X)$.

Throughout this article, we assume that:

(a) firms have no capacity constraint:

$$\forall i \in N,\ q_i = +\infty;$$

(b) firms operate at constant but possibly distinct marginal costs:

$$\forall i \in N, \exists c_i \in \mathbb{R}_+ : C_i(y_i) = c_i y_i,$$

where c_i is firm i's marginal cost, and $y_i \in \mathbb{R}_+$ is the quantity produced by firm i;

(c) firms face the linear inverse demand function:

$$p(X) = a - X,$$

where $X \in \mathbb{R}_+$ is the **total production** of the industry and $a \in \mathbb{R}_+$ is the prohibitive price (the intercept) of the inverse demand function p such that $a \geq 2n \times \max\{c_i : i \in N\}$.

Given assumptions (a), (b) and (c), a Stackelberg oligopoly situation is summarized by the 4-tuple $(L, F, (c_i)_{i\in N}, a)$. Without loss of generality we assume that the firms are ranked according to their marginal costs, i.e., $c_1 \leq \ldots \leq c_n$. For notational convenience, for any set of firms $S \in 2^N \backslash \{\emptyset\}$ we denote the **minimal coalitional cost** by $\underline{c}_S = \min\{c_i : i \in S\}$ and by $i_S \in S$ the firm in S with the smallest index that operates at marginal cost $\underline{c}_S$.

The **strategic Stackelberg oligopoly game** associated with the Stackelberg oligopoly situation $(L, F, (c_i)_{i\in N}, a)$ is a quadruplet $\Gamma_{so} = (L, F, (X_i, \pi_i)_{i\in N})$ defined as:

1. the disjoint finite **sets of leaders and followers** L and F respectively where $N = L \cup F$ is the **set of firms**;
2. for every $k \in N$, an **individual strategy set** X_k such that:
   - for every leader $i \in L$, $X_i = \mathbb{R}_+$ where $x_i \in X_i$ represents the quantity produced by leader i. We denote by $X_L = \prod_{i\in L} X_i$ the set of strategy profiles of the leaders where $x_L = (x_i)_{i\in L}$ is a representative element of X_L;
   - for every follower $j \in F$, X_j is the set of mappings $x_j : X_L \longrightarrow \mathbb{R}_+$ where $x_j(x_L)$ represents the quantity produced by follower j given leaders' strategy profile $x_L \in X_L$. We denote by $X_F = \prod_{j\in F} X_j$ the set of strategy profiles of the followers where $x_F = (x_j)_{j\in F}$ is a representative element of X_F;

3. for every $k \in N$, an **individual profit function** $\pi_k : X_L \times X_F \longrightarrow \mathbb{R}_+$ such that:

   - for every $i \in L$, $\pi_i : X_L \times X_F \longrightarrow \mathbb{R}_+$ is defined as:

   $$\pi_i(x_L, x_F(x_L)) = p(X)x_i - c_i x_i;$$

   - for every $j \in F$, $\pi_j : X_L \times X_F \longrightarrow \mathbb{R}_+$ is defined as:

   $$\pi_j(x_L, x_F(x_L)) = p(X)x_j(x_L) - c_j x_j(x_L),$$

   where $X = \sum_{i \in L} x_i + \sum_{j \in F} x_j(x_L)$ is the **total production**.

Given a strategic Stackelberg oligopoly game $\Gamma_{so} = (L, F, (X_i, \pi_i)_{i \in N})$, every leader $i \in L$ simultaneously and independently produces an output $x_i \in X_i$ at a first period while every follower $j \in F$ simultaneously and independently plays a quantity $x_j(x_L) \in X_j$ at a second period given leaders' strategy profile $x_L \in X_L$. We denote by $\mathcal{G}_{so}$ the **set of strategic Stackelberg oligopoly games**.

In case there is a single leader and multiple followers, Sherali et al. (1983) prove the existence and uniqueness of the Nash equilibrium in strategic Stackelberg oligopoly games under standard assumptions on the inverse demand function and the individual cost functions, i.e., the inverse demand function is twice differentiable, strictly decreasing and satisfies:

$$\forall X \in \mathbb{R}_+, \ \frac{dp}{dX}(X) + X\frac{d^2p}{dX^2}(X) \leq 0,$$

and the individual cost functions are twice differentiable and convex. In particular, they show that the convexity of followers' reaction functions with respect to leader's output is crucial for the uniqueness of the Nash equilibrium. Assumptions (a), (b) and (c) ensure that Sherali et al.'s result (1983) holds on $\mathcal{G}_{so}$ so that any strategic Stackelberg oligopoly game $\Gamma_{so} = (L, F, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{so}$ such that $|L| = 1$ admits a unique Nash equilibrium.

Now, given a strategic Stackelberg oligopoly game $\Gamma_{so} = (L, F, (X_i, \pi_i)_{i \in N}) \in \mathcal{G}_{so}$, we want to associate a Stackelberg oligopoly TU-game in γ-characteristic function form. In a dynamic oligopoly "à la Stackelberg," this assumption implies that the coalition members produce an output at a first period, thus anticipating outsiders' reaction who simultaneously and independently play a quantity at a second period. For any coalition $S \in 2^N \backslash \{\emptyset\}$ where $S = L$ and $N \backslash S = F$, the **coalition profit function** $\pi_S : X_S \times X_{N \backslash S} \longrightarrow \mathbb{R}$ is defined as:

$$\pi_S(x_S, x_{N \backslash S}(x_S)) = \sum_{i \in S} \pi_i(x_S, x_{N \backslash S}(x_S)).$$

Moreover, **followers' individual best reply strategies** $\tilde{x}_{N\backslash S} : X_S \longrightarrow X_{N\backslash S}$ is defined as:

$$\forall j \in N\backslash S, \forall x_S \in X_S,\ \tilde{x}_j(x_S) \in \arg \max_{x_j(x_S)\in X_j} \pi_j(x_S, \tilde{x}_{N\backslash(S\cup\{j\})}(x_S), x_j(x_S)).$$

For any coalition $S \in 2^N\backslash\{\emptyset\}$, given the strategic Stackelberg oligopoly game $\Gamma_{so} = (S, N\backslash S, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{so}$, a strategy profile $(x^*_S, \tilde{x}_{N\backslash S}(x^*_S)) \in X_S \times X_{N\backslash S}$ is a **partial agreement equilibrium** under S if:

$$\forall x_S \in X_S,\ \pi_S(x^*_S, \tilde{x}_{N\backslash S}(x^*_S)) \geq \pi_S(x_S, \tilde{x}_{N\backslash S}(x_S)),$$

and

$$\forall j \in N\backslash S, \forall x_j \in X_j,\ \pi_j(x^*_S, \tilde{x}_{N\backslash S}(x^*_S)) \geq \pi_j(x^*_S, \tilde{x}_{N\backslash(S\cup\{j\})}(x^*_S), x_j(x^*_S)).$$

For any coalition $S \in 2^N\backslash\{\emptyset\}$ and the induced strategic Stackelberg oligopoly game $\Gamma_{so} = (S, N\backslash S, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{so}$, the associated **Stackelberg oligopoly TU-game in γ-characteristic function form**, denoted by (N, v_γ), is defined as:

$$v_\gamma(S) = \pi_S(x^*_S, \tilde{x}_{N\backslash S}(x^*_S)),$$

where $(x^*_S, \tilde{x}_{N\backslash S}(x^*_S)) \in X_S \times X_{N\backslash S}$ is a partial agreement equilibrium under S. We denote by $G^\gamma_{so} \subseteq G$ the **set of Stackelberg oligopoly TU-games in γ-characteristic function form**.

Given a strategic Stackelberg oligopoly game $\Gamma_{so} = (S, N\backslash S, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{so}$, by assumptions (a), (b) and (c), and by the definition of the partial agreement equilibrium, any deviating coalition $S \in 2^N\backslash\{\emptyset\}$ can be represented by firm $i_S \in S$ (the firm in S with the smallest marginal cost) acting as a single leader while the other firms in coalition S play a zero output. It follows from Sherali et al.'s result (1983) that the induced strategic Stackelberg oligopoly game $\Gamma_{so} = (\{i_S\}, N\backslash S, (X_i, \pi_i)_{i\in\{i_S\}\cup N\backslash S}) \in \mathcal{G}_{so}$ has a unique Nash equilibrium, and so the strategic Stackelberg oligopoly game Γ_{so} admits a partial agreement equilibrium under S. Indeed, in case there are at least two firms operating at the minimal marginal cost $\underline{c}_S$, the most efficient firms in coalition S can coordinate their output decision and reallocate the Nash equilibrium output of firm i_S among themselves. We conclude that there can exist several partial agreement equilibria under S which support the unique worth $v_\gamma(S)$. Hence, the γ-characteristic function is well-defined. The following proposition goes further by expressing the worth of any deviating coalition.

Proposition 2.1 *For any coalition $S \in 2^N\backslash\{\emptyset\}$ and the associated strategic Stackelberg oligopoly game $\Gamma_{so} = (S, N\backslash S, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{so}$, it holds that:*

$$v_\gamma(S) = \frac{1}{4(n-s+1)}\left(a + \sum_{j \in N\backslash S} c_j - \underline{c}_S(n-s+1)\right)^2.$$

Proof: Take any coalition $S \in 2^N\backslash\{\emptyset\}$ and consider the associated strategic Stackelberg oligopoly game $\Gamma_{so} = (S, N\backslash S, (X_i, \pi_i)_{i\in N}) \in \mathcal{G}_{so}$. In order to compute the worth $v_\gamma(S)$ of coalition S, we have to successively solve the maximization problems derived from the definition of the partial agreement equilibrium. First, consider the profit maximization program of any follower $j \in N\backslash S$ at the second period:

$$\forall x_S \in X_S, \forall x_{N\backslash(S\cup\{j\})}(x_S) \in X_{N\backslash(S\cup\{j\})}, \quad \max_{x_j(x_S)\in X_j} \pi_j(x_S, x_{N\backslash(S\cup\{j\})}(x_S), x_j(x_S)).$$

The first-order conditions for a maximum are:

$$\forall j \in N\backslash S, \forall x_{N\backslash\{j\}} \in X_{N\backslash\{j\}}, \ \frac{\partial \pi_j}{\partial x_j}(x_j, x_{N\backslash\{j\}}) = 0,$$

and imply that the unique maximizers $\tilde{x}_j(x_S)$, $j \in N\backslash S$, satisfy:

$$\forall j \in N\backslash S, \forall x_S \in X_S, \ \tilde{x}_j(x_S) = \frac{1}{2}\left(a - \sum_{i\in S} x_i - \sum_{k \in N\backslash(S\cup\{j\})} \tilde{x}_k(x_S) - c_j\right).$$

By solving the above system of equations, we deduce that followers' individual best reply strategies at the second period are given by:

$$\forall j \in N\backslash S, \forall x_S \in X_S, \ \tilde{x}_j(x_S) = \frac{1}{(n-s+1)}\left(a - \sum_{i\in S} x_i + \sum_{k\in N\backslash S} c_k\right) - c_j \qquad (2)$$

Then, given $\tilde{x}_{N\backslash S}(x_S) \in X_{N\backslash S}$ consider the profit maximization program of coalition S at the first period:

$$\max_{x_S \in X_S} \pi_S(x_S, \tilde{x}_{N\backslash S}(x_S)).$$

Since the firms have no capacity constraint, it follows that the above profit maximization program of coalition S is equivalent to the profit maximization program of firm $i_S \in S$ given that the other members in S play a zero output:

$$\max_{x_{i_S} \in X_{i_S}} \pi_{i_S}(x_{i_S}, 0_{S\backslash\{i_S\}}, \tilde{x}_{N\backslash S}(x_{i_S}, 0_{S\backslash\{i_S\}})).$$

The first-order condition for a maximum is:

$$\frac{\partial \pi_{i_S}}{\partial x_{i_S}}(x_{i_S}, 0_{S\backslash\{i_S\}}, \tilde{x}_{N\backslash S}(x_{i_S}, 0_{S\backslash\{i_S\}})) = 0,$$

and implies that the unique maximizer $x^*_{i_S} \in X_{i_S}$ is given by:

$$x^*_{i_S} = \frac{1}{2}\left(a + \sum_{j \in N \backslash S} c_j - \underline{c}_S(n-s+1)\right) \tag{3}$$

By (2) and (3), for any $j \in N\backslash S$ it holds that:

$$\begin{aligned} \tilde{x}_j(x^*_S) &= \tilde{x}_j(x^*_{i_S}, 0_{S\backslash\{i_S\}}) \\ &= \frac{1}{2(n-s+1)}\left(a + \sum_{k \in N\backslash S} c_k + \underline{c}_S(n-s+1)\right) - c_j \end{aligned} \tag{4}$$

By (3) and (4), we deduce that:

$$\begin{aligned} v_\gamma(S) &= \pi_S(x^*_S, \tilde{x}_{N\backslash S}(x^*_S)) \\ &= \pi_{i_S}((x^*_{i_S}, 0_{S\backslash\{i_S\}}), \tilde{x}_{N\backslash S}(x^*_{i_S}, 0_{S\backslash\{i_S\}})) \\ &= \frac{1}{4(n-s+1)}\left(a + \sum_{j \in N\backslash S} c_j - \underline{c}_S(n-s+1)\right)^2, \end{aligned}$$

which completes the proof. ■

Thus, the worth of any deviating coalition is increasing with respect to outsiders' marginal costs and decreasing with respect to the smallest marginal cost among its members. Note that the condition $a \geq 2n \times \max\{c_i : i \in N\}$ (assumption (c)) ensures that the equilibrium outputs in (3) and (4) are positive.

3 Characterization of the core and 1-concavity property

In this section, we first introduce the gap function and the 1-concavity property. Generally speaking, we show that the 1-concavity property of the dual game is a necessary and sufficient condition for the core of the initial TU-game to coincide with the set of imputations and to be non-empty. The dual game of a Stackelberg oligopoly TU-game is of great interest since it gives the marginal contribution of followers to join the grand coalition by turning leaders. Based on the description of the γ-characteristic function in a Stackelberg oligopoly TU-game, the aim is to establish the 1-concavity property of its dual game. To this end, we first characterize the core by proving that it always coincides with the set of imputations. Then, we provide a necessary and sufficient condition under which the core is non-empty, and so the dual game is 1-concave. Finally, we prove that this condition depends on the heterogeneity of firms' marginal costs, i.e., the dual game is 1-concave if and only if firms' marginal costs are not too heterogeneous. This last result extends Marini and Currarini's core non-emptiness result (2003) for oligopoly situations.

In the setting of the core, a helpful tool appears to be the so-called gap function $g^v : 2^N \longrightarrow \mathbb{R}$ defined by $g^v(S) = \sum_{i \in S} m_i^v - v(S)$ for all $S \in 2^N \backslash \{\emptyset\}$ with $m_i^v = v(N) - v(N \backslash \{i\})$, where $g^v(\emptyset) = 0$. So, the gap $g^v(S)$ of any coalition S measures how much the coalitional worth $v(S)$ differs from the aggregate allocation based on the individually marginal contributions. The interrelationship between the gap function and the general inclusion $C(N,v) \subseteq CC(N,v)$ is the following equivalence:

$$C(N,v) = CC(N,v) \Longleftrightarrow 0 \leq g^v(N) \leq g^w(S) \text{ for all } S \in 2^N \backslash \{\emptyset\} \tag{5}$$

In words, the core catcher $CC(N,v)$ coincides with the core $C(N,v)$ if and only if the non-negative gap function g^v attains its minimum at the grand coalition N. If the latter property (5) holds, the savings game $(N,v) \in G$ is said to be 1-convex. We arrive at the first corollary.

Corollary 3.1 *Three equivalent statements for any TU-game $(N,v) \in G$ are the following:*

(i) $I(N,v) \neq \emptyset$;

(ii) $v(N) \geq \sum_{i \in N} v(\{i\})$;

(iii) $g^{v^*}(N) \leq 0$.

In fact, the dual game $(N,v^*) \in G$ of any TU-game $(N,v) \in G$ is treated as a cost game such that the core equality $C(N,v^*) = C(N,v)$ holds, on the understanding that the core of any cost game is defined through the reversed inequalities of (1). Thus, $x \in C(N,v^*)$ if and only if $x \in C(N,v)$. As the counterpart to 1-convex TU-games (with non-negative gap functions), we deal with so-called 1-concave TU-games (with non-positive gap functions). A TU-game $(N,v) \in G$ is said to be 1-concave if its non-positive gap function attains its maximum at the grand coalition N:

$$g^v(S) \leq g^v(N) \leq 0 \text{ for all } S \in 2^N \backslash \{\emptyset\} \tag{6}$$

Corollary 3.2 *Three equivalent statements for any TU-game $(N,v) \in G$ are the following:*

(i) The dual game (N,v^*) is 1-concave, that is (6) applied to (N,v^*) holds;

(ii) It holds that:

$$v(N) \geq \sum_{i \in N} v(\{i\}) \text{ and } v(S) \leq \sum_{i \in S} v(\{i\}) \text{ for all } S \in 2^N \backslash \{\emptyset, N\} \tag{7}$$

(iii) $I(N,v) \neq \emptyset$ *and* $C(N,v) = I(N,v)$.

Proof: In view of Corollary 3.1, together with $C(N,v) \subseteq I(N,v)$, it remains to prove the implication $(iii) \Longrightarrow (ii)$. By contraposition, suppose (ii) does not hold in that there exists $S \in 2^N \backslash \{\emptyset, N\}$ with $v(S) > \sum_{i \in S} v(\{i\})$. Define the allocation $x \in \mathbb{R}^n$ by $x_i = v(\{i\})$ for all $i \in S$ and $x_i = v(\{i\}) + (1/(n-s))\big(v(N) - \sum_{j \in N} v(\{j\})\big)$ for all $i \in N \backslash S$. Obviously, $x \in I(N,v) \backslash C(N,v)$. ■

Now, we want to provide a necessary and sufficient condition under which the dual game of a Stackelberg oligopoly TU-game in γ-characteristic function form is 1-concave. In order to establish the 1-concavity of the dual game $(N, v_\gamma^*) \in G$, we want to show (iii) of Corollary 3.2. First, the following proposition provides a characterization of the core.

Proposition 3.3 *Let* $(N, v_\gamma) \in G_{so}^\gamma$ *be a Stackelberg oligopoly TU-game. Then, it holds that:*

$$C(N, v_\gamma) = I(N, v_\gamma).$$

In order to establish the proof of Proposition 3.3 we first need the following lemma. Given a set of marginal costs $\{c_i\}_{i \in N}$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ let $\alpha(S) = \sum_{j \in S \backslash \{i_S\}} (\underline{c}_S - c_j)^2$ and denote by:

$$A_1(S) = \frac{1}{2} \sum_{j \in S \backslash \{i_S\}} \sum_{k \in S \backslash \{i_S\}} (c_j - c_k)^2; \quad B_1(S) = (s-1)\big(\alpha(S) - A_1(S)\big);$$
$$C_1(S) = -(s-1)\big(s\alpha(S) + A_1(S)\big); \qquad D_1(S) = -(s-1)\big(\alpha(S) + A_1(S)\big).$$

We define the functions $f_1 : \mathbb{N} \times 2^N \backslash \{\emptyset\} \longrightarrow \mathbb{R}$ and $f_2 : \mathbb{N} \times 2^N \backslash \{\emptyset\} \longrightarrow \mathbb{R}$ as:

$$f_1(n,S) = 3A_1(S)n^2 + \big(3A_1(S) + 2B_1(S)\big)n + A_1(S) + B_1(S) + C_1(S);$$
$$f_2(n,S) = A_1(S)n^3 + B_1(S)n^2 + C_1(S)n + D_1(S).$$

Lemma 3.4 *Let* $\{c_i\}_{i \in N}$ *be a set of marginal costs. Then, for any* $n \geq 3$ *and any coalition* $S \in 2^N \backslash \{\emptyset\}$ *such that* $s \in \{2, \ldots, n-1\}$ *it holds that (i)* $f_1(n,S) \geq 0$*, and (ii)* $f_2(n,S) \geq 0$.

Proof: First we show point (i). For any $n \geq 3$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s = n-1$ it holds that:

$$\begin{aligned} f_1(n,S) &= (n^2-4)\alpha(S) + (n^2+5n+5)A_1(S) \\ &\geq 0 \end{aligned} \tag{8}$$

Then, we show that for any $n \geq 3$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, n-1\}$, $f_1(n, S) \geq 0$. We proceed by a double induction on the number of firms $n \geq 3$ and the size $s \in \{2, \ldots, n-1\}$ of coalition S respectively.
Initialisation: assume that $n = 3$ and take any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s = 2$. By (8) it holds that $f_1(3, S) \geq 0$.
Induction hypothesis: assume that for any $n \leq k$ and for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, n-1\}$, $f_1(n, S) \geq 0$.
Induction step: we want to show that for $n = k+1$ and for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, k\}$, $f_1(k+1, S) \geq 0$. It follows from (8) that for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s = k$, $f_1(k+1, S) \geq 0$. It remains to show that for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, k-1\}$, $f_1(k+1, S) \geq 0$. Take any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, k-1\}$. Then it follows from the definition of f_1 and the induction hypothesis that:

$$\begin{aligned} f_1(k+1, S) &= f_1(k, S) + 6A_1(S)k + 6A_1(S) + 2B_1(S) \\ &= f_1(k, S) + A_1(S)(6k - 2s + 8) + 2(s-1)\alpha(S) \\ &\geq 0, \end{aligned}$$

which concludes the proof of point (i).
Then, we show point (ii). For any $n \geq 3$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s = n - 1$ it holds that:

$$\begin{aligned} f_2(n, S) &= (n^2 - 3n + 2)\alpha(S) + (n^2 + n + 2)A_1(S) \\ &\geq 0 \end{aligned} \tag{9}$$

Then, we show that for any $n \geq 3$ and any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, n-1\}$, $f_2(n, S) \geq 0$. We proceed by a double induction on the number of firms $n \geq 3$ and the size $s \in \{2, \ldots, n-1\}$ of coalition S respectively.
Initialisation: assume that $n = 3$ and take any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s = 2$. By (9) it holds that $f_2(3, S) \geq 0$.
Induction hypothesis: assume that for any $n \leq k$ and for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, n-1\}$, $f_2(n, S) \geq 0$.
Induction step: we want to show that for $n = k+1$ and for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, k\}$, $f_2(k+1, S) \geq 0$. It follows from (9) that for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s = k$, $f_2(k+1, S) \geq 0$. It remains to show that for any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, k-1\}$, $f_2(k+1, S) \geq 0$. Take any coalition $S \in 2^N \backslash \{\emptyset\}$ such that $s \in \{2, \ldots, k-1\}$. Then it follows from the definitions of f_1 and f_2, the induction hypothesis and point (i) of Lemma 3.4 that:

$$\begin{aligned} f_2(k+1, S) &= f_2(k, S) + f_1(k, S) \\ &\geq 0, \end{aligned}$$

which concludes the proof of point (ii). ■

Now, we are ready to establish the proof of Proposition 3.3 which consists in showing that the first-mover advantage gives too much power to singletons so that the worth of any deviating coalition is less than or equal to the sum of its members' individual worths except for the grand coalition. Given a set of marginal costs $\{c_i\}_{i\in N}$ and any coalition $S \in 2^N\backslash\{\emptyset\}$ we denote by:

$$A_2(S) = \frac{(n-s)(s-1)}{4n(n-s+1)} \text{ (note that for } s \in \{2,\ldots,n-1\},\ A_2(S) > 0);$$

$$B_2(S) = \frac{1}{2n}\sum_{i\in S}\left(\sum_{j\in N\backslash\{i\}} c_j - nc_i\right) - \frac{1}{2(n-s+1)}\left(\sum_{j\in N\backslash S} c_j - \underline{c}_S(n-s+1)\right);$$

$$C_2(S) = \frac{1}{4n}\sum_{i\in S}\left(\sum_{j\in N\backslash\{i\}} c_j - nc_i\right)^2 - \frac{1}{4(n-s+1)}\left(\sum_{j\in N\backslash S} c_j - \underline{c}_S(n-s+1)\right)^2.$$

These quantities will be used in the following proof.

Proof (of Proposition 3.3): First, assume that $n = 2$. By the definitions of the core and the set of imputations it holds that $C(N, v_\gamma) = I(N, v_\gamma)$.
Then, assume that $n \geq 3$. The core is equal to the set of imputations if and only if:

$$\forall S \in 2^N\backslash\{\emptyset\} : s \in \{2,\ldots,n-1\},\ v_\gamma(S) \leq \sum_{i\in S} v_\gamma(\{i\}).$$

In order to prove the above condition, take any coalition $S \in 2^N\backslash\{\emptyset\}$ such that $s \in \{2,\ldots,n-1\}$. By Proposition 2.1 we deduce that:

$$\sum_{i\in S} v_\gamma(\{i\}) - v_\gamma(S) = A_2(S)a^2 + B_2(S)a + C_2(S).$$

Now, we define the mapping $P_S : \mathbb{R} \longrightarrow \mathbb{R}$ as:

$$P_S(y) = A_2(S)y^2 + B_2(S)y + C_2(S),$$

so that $P_S(a) = \sum_{i\in S} v_\gamma(\{i\}) - v_\gamma(S)$. We want to show that for any $y \in \mathbb{R}$, $P_S(y) \geq 0$. It follows from $A_2(S) > 0$ that the minimum of P_S is obtained at point $y^* \in \mathbb{R}$ such that:

$$y^* = -\frac{B_2(S)}{2A_2(S)}.$$

After some calculation steps, we obtain that the minimum of P_S is equal to:

$$P_S(y^*) = \frac{1}{4n(n-s)(s-1)} f_2(n, S),$$

where f_2 is defined as in Lemma 3.4. Hence, it follows from point (ii) of Lemma 3.4 that $P_S(y^*) \geq 0$, which implies that for any $y \in \mathbb{R}$, $P_S(y) \geq 0$. In particular, we conclude that $P_S(a) \geq 0$, and so $\sum_{i \in S} v_\gamma(\{i\}) - v_\gamma(S) \geq 0$. ■

Now, we provide a necessary and sufficient condition for the non-emptiness of the core of Stackelberg oligopoly TU-games in γ-characteristic function form as enunciated in the following proposition.

Theorem 3.5 *Let $(N, v_\gamma) \in G_{so}^\gamma$ be a Stackelberg oligopoly TU-game. Then, its dual game $(N, v_\gamma^*) \in G$ is 1-concave if and only if:*

$$2a\left(\sum_{i \in N} c_i - n\underline{c}_N\right) \geq \sum_{i \in N}\left(\sum_{j \in N \setminus \{i\}} c_j - nc_i\right)^2 - n\underline{c}_N^2 \tag{10}$$

or equivalently

$$2a(\overline{c}_N - \underline{c}_N) \geq \frac{(n+1)^2}{n}\sum_{j \in N} c_j^2 - \frac{(n+2)}{n}\left(\sum_{j \in N} c_j\right)^2 - \underline{c}_N^2 \tag{11}$$

where $\overline{c}_N = \sum_{i \in N} c_i/n$ is the average cost of the grand coalition.

Proof: It follows from Proposition 3.3 that the core is non-empty if and only if $\sum_{i \in N} v_\gamma(\{i\}) \leq v_\gamma(N)$. By Proposition 2.1 it holds that:

$$\begin{aligned}
\sum_{i\in N} v_\gamma(\{i\}) &= \frac{1}{4n}\sum_{i\in N}\left(a+\sum_{j\in N\setminus\{i\}} c_j - nc_i\right)^2\\
&= \frac{1}{4n}\sum_{i\in N}\left(a+\sum_{j\in N\setminus\{i\}} c_j - nc_i - \underline{c}_N + \underline{c}_N\right)^2\\
&= \frac{1}{4n}\sum_{i\in N}\left[\left(a-\underline{c}_N\right)^2 + 2(a-\underline{c}_N)\left(\sum_{j\in N\setminus\{i\}} c_j - nc_i + \underline{c}_N\right)\right.\\
&\qquad\qquad\qquad\left. + \left(\sum_{j\in N\setminus\{i\}} c_j - nc_i + \underline{c}_N\right)^2\right]\\
&= v_\gamma(N) + \frac{1}{4n}\left[2a\left(n\underline{c}_N - \sum_{i\in N} c_i\right) + \sum_{i\in N}\left(\sum_{j\in N\setminus\{i\}} c_j - nc_i\right)^2 - n\underline{c}_N^2\right]\\
&= v_\gamma(N) + \frac{1}{4n}\left[2a\left(n\underline{c}_N - n\bar{c}_N\right) + (n+1)^2\sum_{j\in N} c_j^2\right.\\
&\qquad\qquad\qquad\left. - (n+2)\left(\sum_{j\in N} c_j\right)^2 - n\underline{c}_N^2\right],
\end{aligned}$$

which permits to conclude that $\sum_{i\in N} v_\gamma(\{i\}) \leq v_\gamma(N)$ if and only if inequalities (10) or (11) holds. ■

In case all the firms have the same marginal cost, the both sides of inequality (10) are equal to zero which implies that $\sum_{i\in N} v_\gamma(\{i\}) = v_\gamma(N)$, and so $(v_\gamma(\{i\}))_{i\in N} = (v_\gamma(N)/n)_{i\in N}$ is the unique core element which coincides with Marini and Currarini's core allocation result (2003).

Driessen et al. (2010) show that the nucleolus of any 1-concave TU-game coincides with the center of gravity of the core. Hence, for any Stackelberg oligopoly TU-game $(N, v_\gamma) \in G^\gamma_{so}$, the dual game (N, v^*_γ) is 1-concave if and only if inequality (10) holds, implying that the nucleolus of (N, v^*_γ) is the center of gravity of the core of (N, v^*_γ).

The following theorem gives a more relevant expression of inequality (10). When the difference between any two successive marginal costs is constant, it provides an upper bound on the heterogeneity of firms' marginal costs below which inequality (10) holds.

Theorem 3.6 *Let $(N, v_\gamma) \in G^\gamma_{so}$ be a Stackelberg oligopoly TU-game such that:*

$$\exists \delta \in \mathbb{R}_+ : \forall i \in \{1, \ldots, n-1\},\ c_{i+1} = c_i + \delta \tag{12}$$

Then, inequality (10) *holds if and only if:*

$$\delta \leq \delta^*(n) = \frac{12(a - \underline{c}_N)(n-1)}{n^4 + 2n^3 + 3n^2 - 8n + 2} \tag{13}$$

Proof: It follows from (12) that inequality (10) can be expressed as:

$$\sum_{i \in N} \left(\delta \frac{(n^2 - 2ni + n)}{2} - c_i \right)^2 - n\underline{c}_N^2 \leq an(n-1)\delta.$$

By noting that:

$$\sum_{i \in N} c_i^2 - n\underline{c}_N^2 = \delta \underline{c}_N n(n-1) + \delta^2 \frac{n(n-1)(2n-1)}{6},$$

we deduce that the above inequality is equivalent to:

$$\delta \left[\sum_{i \in N} \left(\frac{n^2 - 2ni + n}{2} \right)^2 + \frac{n(n-1)(2n-1)}{6} \right] \leq (a - \underline{c}_N)n(n-1) + \sum_{i \in N} (n^2 - 2ni + n)c_i \tag{14}$$

It remains to compute the two sums in inequality (14). First, it holds that:

$$\begin{aligned} \sum_{i \in N} \left(\frac{n^2 - 2ni + n}{2} \right)^2 &= \frac{n^3(n+1)^2}{4} - n^2(n+1) \sum_{i \in N} i + n^2 \sum_{i \in N} i^2 \\ &= \frac{1}{12} \big(2n^4(n+1) - n^3(n+1)^2 \big) \end{aligned} \tag{15}$$

Then, it holds that:

$$\begin{aligned} \sum_{i \in N} c_i &= n\underline{c}_N + \delta \sum_{i=1}^{n-1} i \\ &= n \left(\underline{c}_N + \delta \frac{(n-1)}{2} \right), \end{aligned}$$

and

$$\begin{aligned} \sum_{i \in N} ic_i &= \underline{c}_N \sum_{i=1}^{n} i + \delta \sum_{i=1}^{n} i(i-1) \\ &= \underline{c}_N \frac{n(n+1)}{2} + \delta \frac{n(n+1)(2n-2)}{6}. \end{aligned}$$

Hence, we deduce that:

$$\begin{aligned}\sum_{i\in N}(n^2-2ni+n)c_i &= (n^2+n)\sum_{i\in N}c_i - 2n\sum_{i\in N}ic_i \\ &= \frac{1}{6}\big(\delta n^2(n+1)(1-n)\big) \qquad (16)\end{aligned}$$

By (15) and (16), we conclude that inequality (14) is equivalent to (13). ■

By noting that:

$$\begin{aligned}\frac{d\delta^*}{dn}(n) &= -\frac{36(a-\underline{c}_N)(n^2+2n+2)}{(n^3+3n^2+6n-2)^2} \\ &< 0,\end{aligned}$$

and

$$\begin{aligned}\frac{d^2\delta^*}{dn^2}(n) &= \frac{72(a-\underline{c}_N)(2n^4+8n^3+15n^2+20n+14)}{(n^3+3n^2+6n-2)^3} \\ &> 0,\end{aligned}$$

we deduce that the bound $\delta^*(n)$ is strictly decreasing and strictly convex with respect to the number of firms n. Moreover, when n tends to infinity its limit is equal to 0. So, the more the number of firms is, the less the heterogeneity of firms' marginal costs must be in order to ensure the non-emptiness of the core. This result extends Marini and Currarini's core allocation result (2003) and shows that their result crucially depends on the symmetric players assumption.

We saw that when the heterogeneity of firms' marginal costs increases the core becomes smaller. Surprisingly, in case the inverse demand function is strictly concave, the following example shows that the opposite result may holds, i.e., when the heterogeneity of firms' marginal costs increases the core becomes larger.

Example 3.7

Consider the three Stackelberg oligopoly TU-games $(N, v^1_\gamma) \in G^\gamma_{so}$, $(N, v^2_\gamma) \in G^\gamma_{so}$ and $(N, v^3_\gamma) \in G^\gamma_{so}$ associated with the Stackelberg oligopoly situations $(L, F, (c_1, c^1_2), p)$, $(L, F, (c_1, c^2_2), p)$ and $(L, F, (c_1, c^3_2), p)$ respectively where $N = \{1, 2\}$, $c_1 = c^1_2 = 2$, $c^2_2 = 4$, $c^3_2 = 5$ and $p = 10 - X^2$. The worths of any coalition $S \in 2^N\backslash\{\emptyset\}$ are given in the following table:

S	$\{1\}$	$\{2\}$	$\{1,2\}$
$v^1_\gamma(S)$	4.70	4.70	8.71
$v^2_\gamma(S)$	6.19	2.02	8.71
$v^3_\gamma(S)$	7.05	1.05	8.71

Thus, it holds that $\emptyset = C(N, v_\gamma^1) \subset C(N, v_\gamma^2) \subset C(N, v_\gamma^3)$, and so when the heterogeneity of firms' marginal costs increases the core becomes larger. $\square$

4 Concluding remarks

In this article we have considered Stackelberg oligopoly TU-games in γ-characteristic function form (Chander and Tulkens 1997) in which every deviating coalition produces an output at a first period and outsiders simultaneously and independently play a quantity at a second period. We have assumed that the inverse demand function is linear and that firms operate at constant but possibly distinct marginal and average costs. Thus, contrary to Marini and Currarini (2003) the payoff (profit) functions are not necessarily identical. Generally speaking, for any TU-game we have showed that the 1-concavity property of its dual game is a necessary and sufficient condition under which the core of the initial TU-game coincides with the set of imputations and is non-empty. Particularly, the nucleolus of such a 1-concave dual game agrees with the center of gravity of the set of imputations. The dual game of a Stackelberg oligopoly TU-game is of great interest since it describes the marginal contribution of followers to join the grand coalition by turning leaders. Based on the description of the γ-characteristic function in a Stackelberg oligopoly TU-game, the aim was to establish the 1-concavity property of its dual game. To this end, we have first proved that the core always coincides with the set of imputations. Indeed, the first-mover advantage gives too much power to singletons so that the worth of every deviating coalition is less than or equal to the sum of its members' individual worths except for the grand coalition. Then, we have provided a necessary and sufficient condition under which the core is non-empty. Finally, we have proved that this condition depends on the heterogeneity of firms' marginal costs, i.e., the dual game is 1-concave if and only if firms' marginal costs are not too heterogeneous. The more the number of firms, the less the heterogeneity of firms' marginal costs must be in order to ensure the 1-concavity property of the dual game. This last result extends Marini and Currarini's core non-emptiness result (2003) for oligopoly situations and shows that their result crucially depends on their symmetric players assumption. Surprisingly, in case the inverse demand function is strictly concave, we have provided an example in which the opposite result holds, i.e., when the heterogeneity of firms' marginal costs increases the core becomes larger.

Instead of quantity competition, we can associate a two-stage structure with the γ-characteristic function in a price competition setting. Marini and Currarini's core non-emptiness result (2003) applies to this setting and they provide examples in which the core of the sequential Bertrand oligopoly TU-games is included in the core of the classical Bertrand oligopoly TU-games. A question concerns the effect of the heterogeneity of firms' marginal costs on the non-emptiness of the core. This is left for future work.

References

Aumann, R. (1959). Acceptable points in general cooperative n-person games, in: Tucker, luce (eds.), Contributions to the theory of games IV. *Annals of Mathematics Studies Vol. 40*, Princeton University Press, Princeton.

Branzei, R., D. Dimitrov, and S. Tijs (2003). Shapley-like values for interval bankruptcy games. *Economics Bulletin 3*, 1–8.

Chander, P. and H. Tulkens (1997). The core of an economy with multilateral environmental externalities. *International Journal of Game Theory 26*, 379–401.

Driessen, T. S., A. Khmelnitskaya, and P. Dehez (2010). One-concavity of data cost games. *Memorandum.*

Driessen, T. S. and H. I. Meinhardt (2005). Convexity of oligopoly games without transferable technologies. *Mathematical Social Sciences 50*, 102–126.

Lardon, A. (2009). The γ-core of cournot oligopoly games with capacity constraints. *Forthcoming in Theory and Decision, DOI 10.1007/s11238-011-9256-5*.

Lardon, A. (2010). Cournot oligopoly interval games. Working paper, University of Saint-Etienne.

Marini, M. A. and S. Currarini (2003). A sequential approach to the characteristic function and the core in games with externalities. *Advances in Economic Design, Springer Verlag*, 233–251.

Norde, H., K. H. Pham Do, and S. Tijs (2002). Oligopoly games with and without transferable technologies. *Mathematical Social Sciences 43*, 187–207.

Sherali, H. D., A. L. Soyster, and F. H. Murphy (1983). Stackelberg-Nash-Cournot Equilibria: Characterizations and Computations. *Operations Research 31*(2), 253–276.

Zhao, J. (1999b). A β-core existence result and its application to oligopoly markets. *Games and Economic Behavior 27*, 153–168.

Zhao, J. (1999a). A necessary and sufficient condition for the convexity in oligopoly games. *Mathematical Social Sciences 37*, 189–204.